Neumann120

von Neumann and rings of operators

Alain Connes

- Rings of operators.

- The intrinsic time of a factor.

- Classification of factors.

- Foliations and noncommutative spaces.

- The birth of noncommutative geometry

- The new calculus and geometry.

3

Factorizations

Let the Hilbert space $\mathcal{H}$ factor as a tensor product :

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Von Neumann, with his post-doctoral student Murray, investigated the meaning of such a factorization at the level of operators.

A factor is an algebra of operators which has all the obvious properties of the algebra of operators of the form $T_1 \otimes 1$ acting in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

4. Another interpretation of $(\overline{\mathrm{D}}_5)$ is suggested by quantum mechanics. The operators of $\mathfrak{H}$ correspond there to all observable quantities which occur in a mechanical system $\mathfrak{S}$. (Cf. (6), pp. 55–60, and (2C), p. 167. We restrict ourselves to bounded operators, which correspond to those observables which have a bounded range. Thus $\mathbf{B}$ corresponds to the totality of these observables.) Now if $\mathfrak{S}$ can be decomposed into two parts $\mathfrak{S}_1$, $\mathfrak{S}_2$ and if we denote the set of the operators which correspond to observables situated entirely in $\mathfrak{S}_1$ or in $\mathfrak{S}_2$ by $\mathbf{M}_1$ resp. $\mathbf{M}_2$, then we see:

(1) $\mathbf{M}_1$, $\mathbf{M}_2$ are rings, and 1 (which corresponds to the "constant" observable 1) belongs to both $\mathbf{M}_1$, $\mathbf{M}_2$.

(2) If $A \in \mathbf{M}_1$, $B \in \mathbf{M}_2$ then the measurements of the observables of A and B do not interfere (being in different parts of $\mathfrak{S}$); therefore A, B commute (cf. (6), pp. 11–14 and 76, or (20), pp. 117–121). Thus $\mathbf{M}_2 \subset \mathbf{M}_1'$.

(3) As $\mathfrak{S}$ is the sum of $\mathfrak{S}_1$, $\mathfrak{S}_2$ therefore $\mathbf{R}(\mathbf{M}_1, \mathbf{M}_2) = \mathbf{B}$.

Thus our problem of solving $(\overline{D}_5)$ corresponds to the quantum mechanical problem of dividing a system $\mathfrak{S}$ into two subsystems $\mathfrak{S}_1$, $\mathfrak{S}_2$; and in particular the solutions **M** of $(\overline{D}_5)$ correspond to the complete rings of all observables of suitable quantum mechanical systems.

This interpretation of $(\overline{D}_5)$ suggests of course strongly the surmise formulated at the end of §2.2: It should be possible to describe $\mathfrak{H}$ as (isomorphic to) the space of all two variable functions $f(x, y)$, ($\iint |f(x, y)|^2\, dx\, dy$ finite), **M** operating on x only, and **M**$'$ on y only. In this case $\mathfrak{S}_1$, $\mathfrak{S}_2$ would be explicitly given: $\mathfrak{S}_1$ being described by the coordinate x, and $\mathfrak{S}_2$ by the coordinate y.

The fact that the surmise of §2.2 is not true, is therefore the more remarkable; particularly so because certain features of the "exceptional" rings **M** seem to make them even better suited for quantum mechanical purposes than the customary **B**. We will now discuss these properties of **M**.

Three types

Type I, if the Hilbert space $\mathcal{H}$ factors as a tensor product :

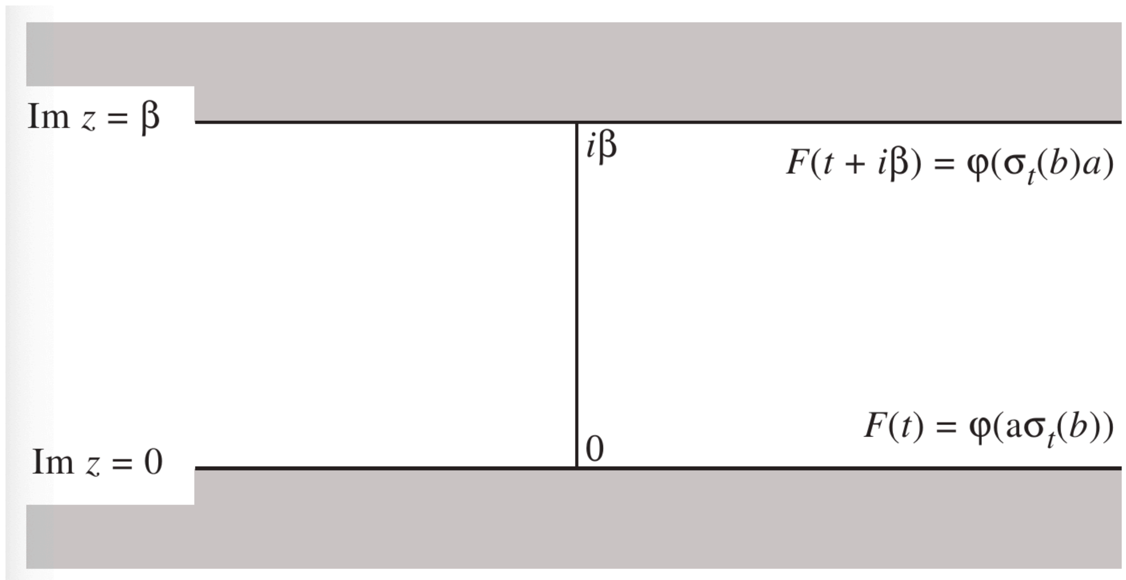
$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Von Neumann and Murray found two other types :

Type II : The classification of subspaces gives an interval $[0, 1]$ or $[0, \infty]$; continuous dimensions ! Trace erases noncommutativity.

Type III : All that remains.

Quantum Statistical mechanics, KMS Condition



Haag, Hugenholtz and Winnink, time and thermodynamics, link between tr$(A\exp(-\beta H))$ and the Heisenberg evolution $\sigma_t(A) = \exp(itH)A\exp(-itH)$. $\beta = \hbar/kT$.

Tomita–Takesaki, 1970

Theorem

Let M be a von Neumann algebra and φ a faithful normal state on M, then there exists a unique

$$\sigma_t^\varphi \in \mathrm{Aut}(M)$$

which fulfills the KMS condition for $\beta = 1$.

Thesis (1971–1972)

Theorem (ac)

$$1 \to \mathrm{Int}(\mathcal{M}) \to \mathrm{Aut}(\mathcal{M}) \to \mathrm{Out}(\mathcal{M}) \to 1,$$

The class of σ_t^φ in $\mathrm{Out}(\mathcal{M})$ does not depend on φ.

Thus a von Neumann algebra $\mathcal{M}$, has a canonical evolution

$$\mathbb{R} \xrightarrow{\delta} \mathrm{Out}(\mathcal{M}).$$

Noncommutativity $\Rightarrow$ Evolution

Classification of factors

New invariants and reduction of type III to type II and automorphisms were done in my thesis.

The Module $S(M)$: It is a closed subgroup of $\mathbb{R}_+^*$,

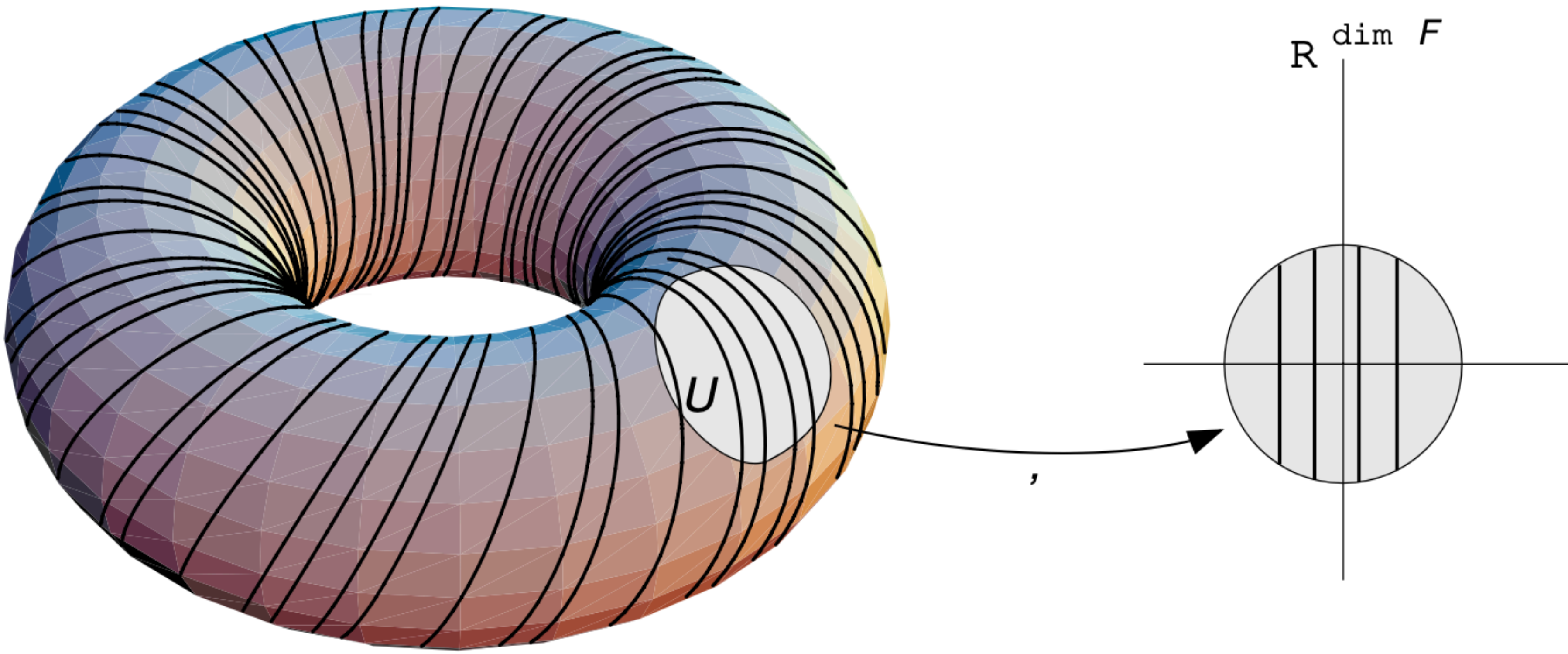
Factors of type III$_\lambda$, $\lambda \in [0,1]$

Periods : $T(M) \subset \mathbb{R}$ is the subgroup of $\mathbb{R}$ kernel of the time evolution.

Classification of hyperfinite factors

(ac 1976)

The von Neumann algebra of a foliation



Random operators

They form a von Neumann algebra canonically associated to the foliation

$$T_\ell \quad \textit{operator in } L^2(\ell)$$

▶ Measure theory = von Neumann algebras.

▶ Topology = C^*-algebras.

▶ Geometry = Spectral triples.

▶ Quantized Calculus.

Continuous and discrete

Classical formulation of real variables :

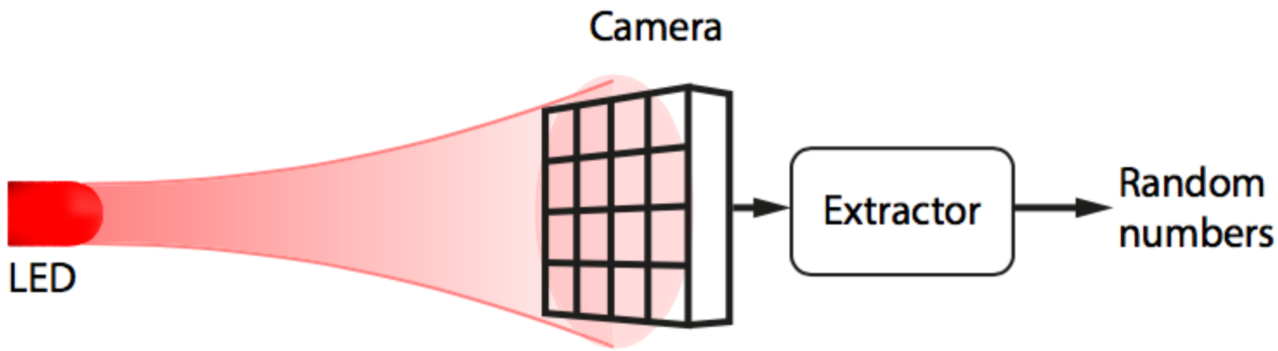
$$f : X \to \mathbb{R}$$

Discrete and continuous variables cannot coexist in this classical formalism. Solved using the formalism of quantum mechanics created by John von Neumann.

Classical	**Quantum**
Real variable $f : X \to \mathbb{R}$	Self-adjoint operator in Hilbert space
Possible values of the variable	Spectrum of the operator
Algebraic operations on functions	Algebra of operators in Hilbert space

Quantum variability

Quantum random number generation on a mobile phone



Newton

“In a certain problem, a variable is the quantity that takes an infinite number of values which are quite determined by this problem and are arranged in a definite order"

“A variable is called infinitesimal if among its particular values one can be found such that this value itself and all following it are smaller in absolute value than an arbitrary given number"

Infinitesimal variables

What is surprising is that the Quantum set-up immediately provides a natural home for the “infinitesimal variables" and here the distinction between “variables" and numbers (in many ways this is where the point of view of Newton is more efficient than that of Leibniz) is essential.

Classical	**Quantum**
Infinitesimal variable	Compact operator in Hilbert space
Infinitesimal of order α	$\mu_n(T)$ of size $n^{-\alpha}$ when $n \to \infty$
Integral of function $\int f(x)dx$	$\displaystyle{-}\!\!\!\!\!\!\int T$ = coefficient of $\log(\Lambda)$ in $\mathrm{Tr}_\Lambda(T)$

Geometry from the spectral point of view

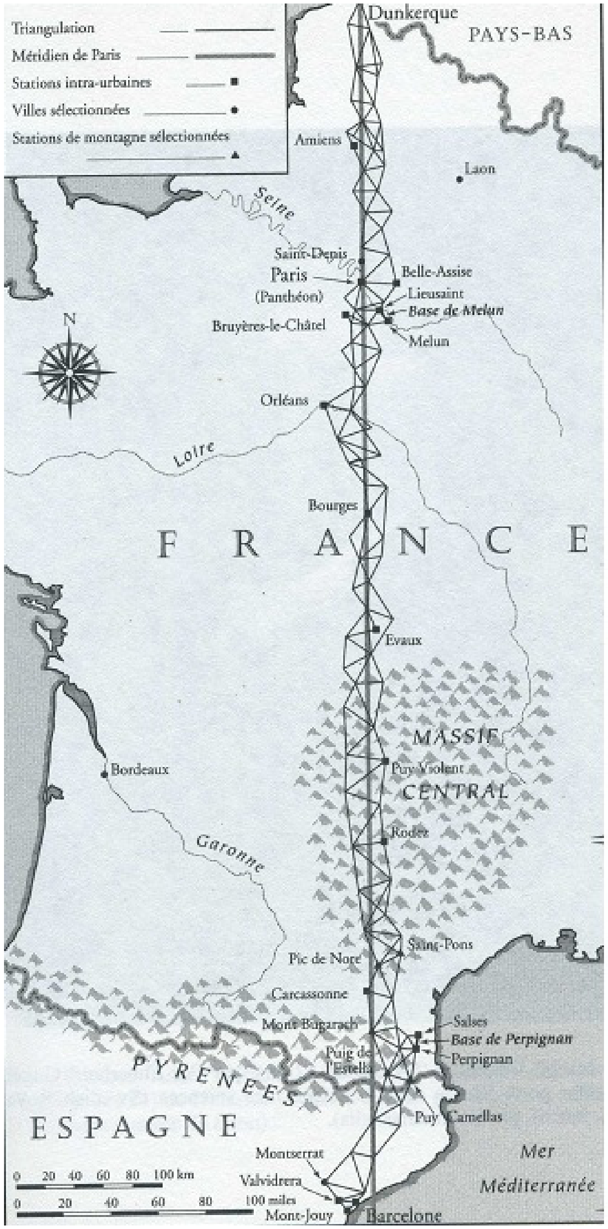
Es muss also entweder das dem Raume zu Grunde liegende Wirkliche eine discrete Mannigfaltigkeit bilden, oder der Grund der Massverhältnisse ausserhalb, in darauf wirkenden bindenen Kräften, gesucht werden.

Either therefore the reality which underlies space must be discrete, or we must seek the foundation of its metric relations outside it, in binding forces which act upon it.

Line element

The Riemannian paradigm is based on the Taylor expansion in local coordinates of the square of the line element and in order to measure the distance between two points one minimizes the length of a path joining the two points

$$d(a,b) = \mathrm{Inf} \int_\gamma \sqrt{g_{\mu\,\nu}\,dx^\mu\,dx^\nu}$$



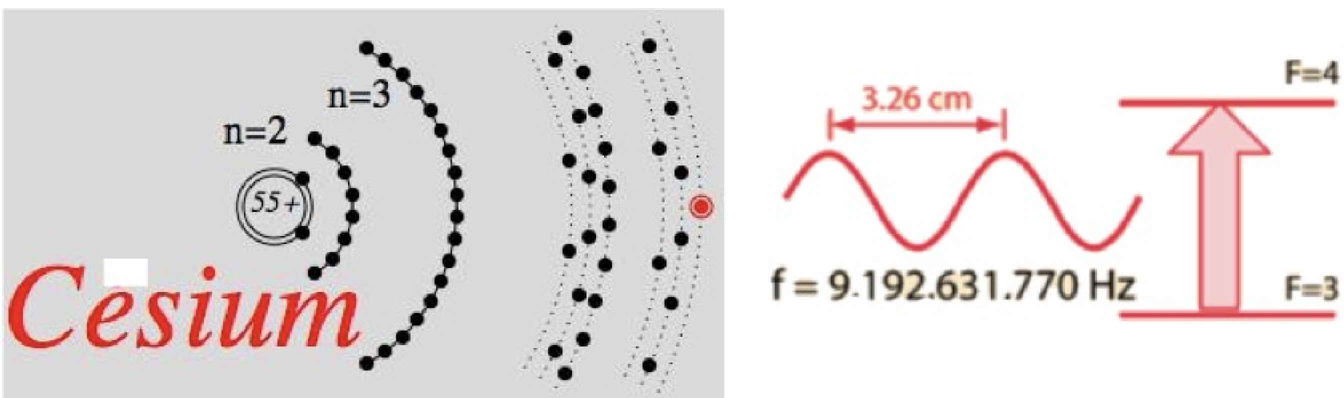
J-B. J. DELAMBRE

P. F. A. MECHAIN

1792--1799

DUNKERQUE--BARCELONE

Change of unit of length, 1967, 1984



Meter $\rightarrow$ Wave length (Krypton (1967) spectrum of 86Kr then Caesium (1984) hyperfine levels of C133)

Spectral paradigm

P. Dirac showed how to extract the square root of the Laplacian and this provides a direct connection with the quantum formalism : the line element is the propagator

$$ds = D^{-1}$$

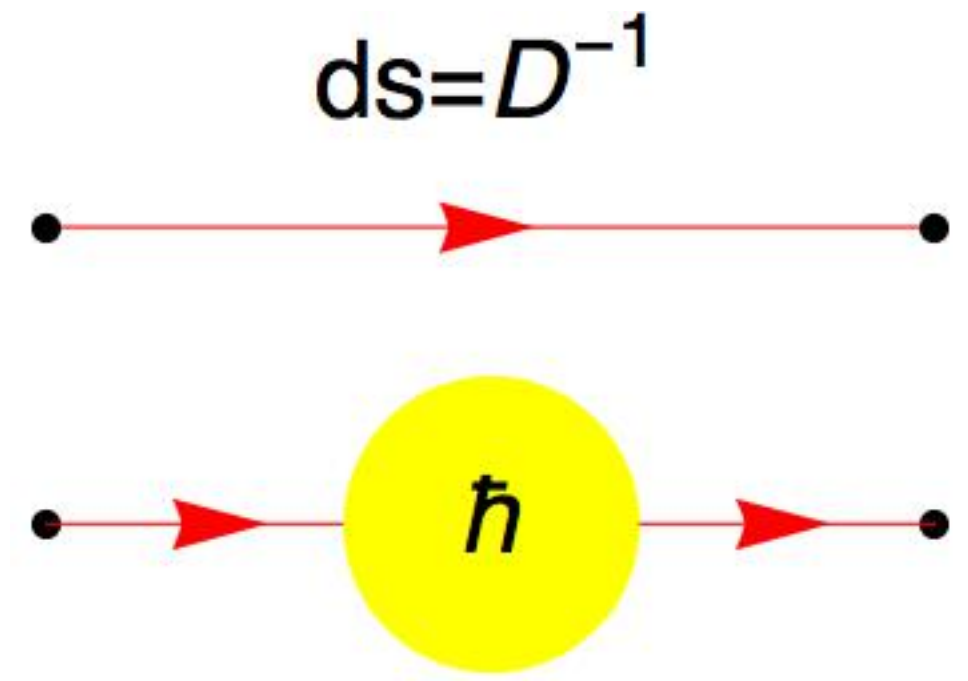
$$d(a,b) = \text{Sup}\,|f(a) - f(b)| \mid \|[D,f]\| \leq 1.$$

This is a “Kantorovich dual" of the usual formula.

Classical	**Quantum**
Line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$	Propagator operator $ds := D^{-1}$
$d(a,b) =$ Inf $\int_\gamma \sqrt{ds^2}$	$d(a,b) =$ Sup $\lvert f(a) - f(b) \rvert$ $\lVert [D,f] \rVert \leq 1$
Volume $\int \sqrt{g} d^4 x$	$⨍ ds^4 =$ coefficient of $\log(\Lambda)$ in $\mathrm{Tr}_\Lambda(ds^4)$

Line element

The line element contains all the information of the gauge potentials *i.e.* of the forces binding the space together, moreover it is dressed by the quantum corrections.



Pure gravity

In our joint work with A. Chamseddine, W. van Suijlekom, we express the very elaborate Lagrangian given by gravity coupled with the Standard Model, with all its subtleties (V-A, BEH, seesaw, etc etc...) as pure gravity on a geometric space-time whose texture is slightly more elaborate than the 4-dimensional continuum.

- How to get the algebra : Clifford + punctuation

- How to get the Einstein action : Spectral Action

Language $\Rightarrow$ why NC simplifies

The language respects NC and is a much more informative datum, with $M_2(\mathbb{C})$ and Y, $Y^2 = 1$ one generates all matrix valued functions on the two-sphere.
One obtains all spin 4-manifolds using the slight amount of noncommutativity provided by

$$M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

which appear as Clifford algebras and give the SM gauge group.

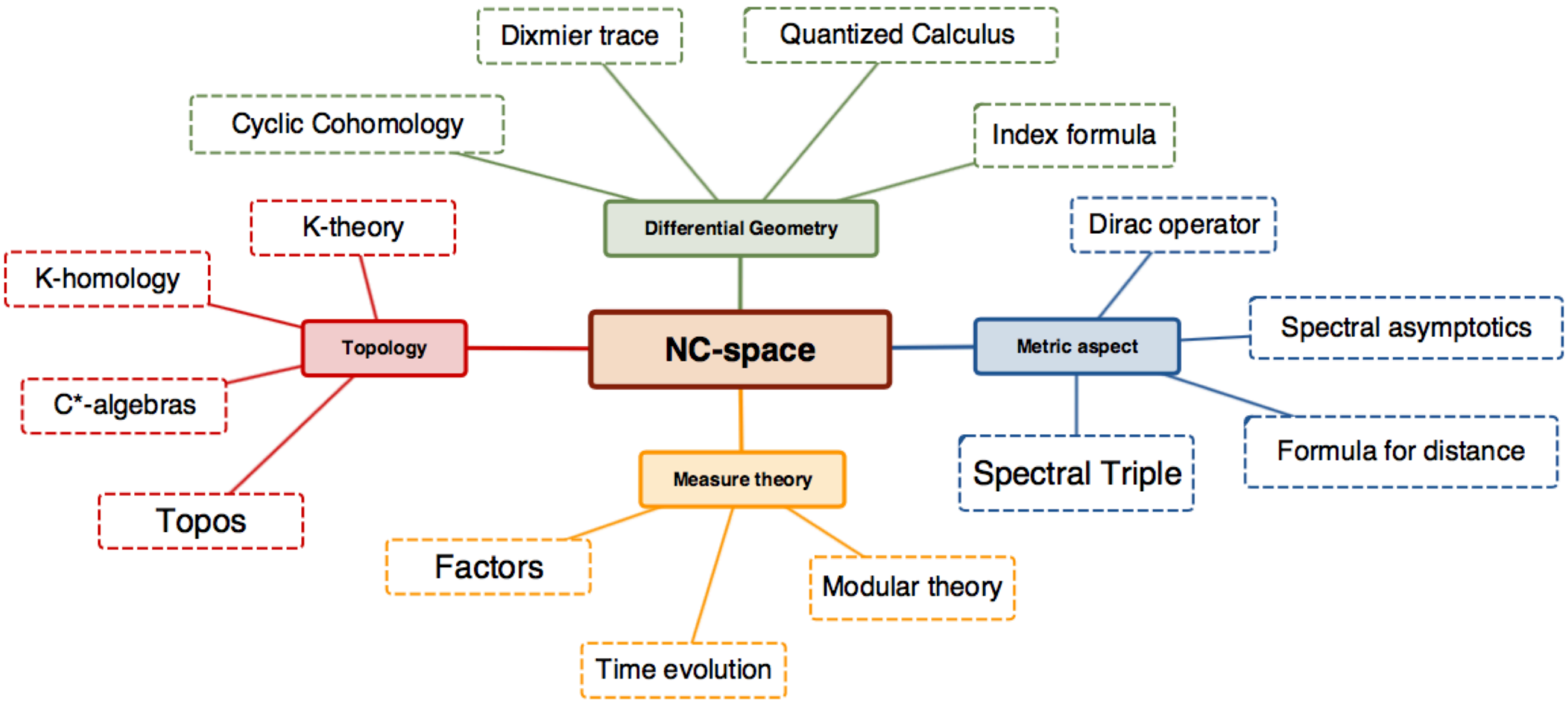
Higher Heisenberg equation

One introduces a variable Y with $Y^4 = 1$, and the quantization condition takes J and γ into account :

$$\frac{1}{n!}\langle Z\,[D,Z]\cdots[D,Z]\rangle = \gamma \quad Z = 2EJEJ^{-1} - 1,$$

$$E = \frac{1}{2}(1 + Y_+) \oplus \frac{1}{2}(1 + iY_-)$$

$$Y = Y_+ \oplus Y_- \in C^\infty(M, C_+ \oplus C_-)$$


Dixmier trace
Quantized Calculus
Cyclic Cohomology
Index formula
Differential Geometry
K-theory
K-homology
Dirac operator
Topology
NC-space
Metric aspect
Spectral asymptotics
C*-algebras
Spectral Triple
Formula for distance
Measure theory
Topos
Factors
Modular theory
Time evolution