Estimates for L-functions, I

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Riemann zeta function (1859)

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Meromorphic continuation.
Functional equation: $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ invariant by $s \mapsto 1 - s$.
Controls distribution of primes, e.g., prime number theorem

$$\lim_{x\to\infty} \frac{\#\{p \leqslant x\}}{x/\log x} = 1$$

"equivalent" to $\zeta(s) = 0 \implies \mathrm{Re}(s) < 1$.
Open problems:

- Riemann hypothesis: $\zeta(s) = 0 \implies \mathrm{Re}(s) \leqslant 1/2$
- Lindelöf hypothesis: $\zeta(1/2 + it) \ll_\varepsilon (1 + |t|)^\varepsilon$.

L-functions

The known generalizations of $\zeta(s)$ are "L-functions of degree n":

$$L(s) = \prod_p \frac{1}{1 - \alpha_{p,1} p^{-s}} \cdots \frac{1}{1 - \alpha_{p,n} p^{-s}}.$$

No standard definition of "*L*-function," but many examples:

- $\zeta(s)$ is an *L*-function of degree 1
- Dirichlet *L*-functions $L(\chi, s)$, attached to Dirichlet characters
- Artin *L*-functions $L(\rho, s)$, attached to Galois representations
- Hasse–Weil zeta functions, attached to varieties
- Hecke *L*-functions, attached to classical modular forms
- Langlands *L*-functions, attached to automorphic forms on reductive groups
- Standard *L*-functions $L(\pi, s)$ of degree n, attached to automorphic forms π on GL_n. Example:

$$\zeta(s)^n \longleftrightarrow \text{Eisenstein series}$$

Conjectures of Langlands, mostly wide open:

- "Every L-function" is a standard L-function.
- L-functions preserved under natural operations, e.g., "tensor product:"

$$L(\pi, s) = \prod_{p,j} (1 - \alpha_{p,j} p^{-s})^{-1}, \quad L(\sigma, s) = \prod_{p,k} (1 - \beta_{p,k} p^{-s})^{-1}$$

$$\rightsquigarrow^{?} L(\pi \times \sigma, s) = \prod_{p,j,k} (1 - \alpha_{p,j} \beta_{p,k} p^{-s})^{-1}$$

Why care about estimating L-functions?

Motivated by questions/applications discovered in the 1980's and 1990's. [1] [2] [3]

1. heuristics for moments of families:

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt \sim_? a_k g_k T(\log T)^{k^2}, \quad (g_k) = 1, 2, 42, 24024, \ldots$$

2. distribution of integral solutions to $n = \square + \square + \square$

$$\longleftrightarrow \text{ nontrivial bounds for } L(\varphi \times \theta_Q, 1/2).$$

3. arithmetic quantum unique ergodicity: $|\varphi_j|^2\,d\mu \sim (?)$,

$$\longleftrightarrow \text{ nontrivial bounds for } L(\varphi_j \times \varphi_j \times \Psi, 1/2).$$

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[1] Keating–Snaith, Conrey–Farmer–Keating–Rubinstein-Snaith, Diaconu–Goldfeld–Hoffstein, . . .

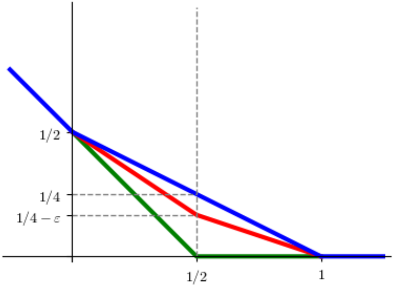
[2] Iwaniec, Duke, Duke–Schulze-Pillot, Duke–Friedlander–Iwaniec, . . .

[3] Rudnick–Sarnak, Lindenstrauss, Holowinsky–Soundararajan, . . .

Bounds for $\zeta(s)$

Consider $e(\sigma) := \inf\{e \in \mathbb{R} : \zeta(\sigma + it) \ll_\sigma (1 + |t|)^e\}$.

Convexity bound: $e(\frac{1}{2}) \leqslant 1/4$.
Lindelöf hypothesis: $e(\frac{1}{2}) = 0$.
Subconvexity: $e(\frac{1}{2}) < 1/4$.



Weyl, Hardy–Littlewood (1921): $e(\frac{1}{2}) \leqslant 1/6 < 1/4$, i.e.,

$$\zeta(1/2 + it) \ll_\varepsilon (1 + |t|)^{1/6+\varepsilon}.$$

This is the prototypical subconvex bound or "weak Lindelöf hypothesis."

t-aspect subconvexity problem for standard L-functions

For $L(\pi, s)$: standard L-function of degree n, convexity bound reads

$$L(\pi, 1/2 + it) \ll_{\pi,\varepsilon} (1 + |t|)^{n/4+\varepsilon}.$$

Until recently, subconvex bounds known only for $n \leqslant 3$:

- $n = 1$: Weyl/Hardy–Littlewood 1916–1921, ..., Bourgain 2017
- $n = 2$: Good 1982, ..., Duke–Friedlander–Iwaniec 1990's, Michel–Venkatesh 2010, ...
- $n = 3$: Li 2011, Munshi 2015, ..., Blomer–Buttcane 2020, ...
- $\log^{-\delta}$ savings: Soundararajan 2010, Soundararajan–Thorner 2019

Theorem 1 (N; arXiv, October 2021)

For any standard L-function of degree n, we have

$$\boxed{L(\pi, 1/2 + it) \ll_{\pi} (1 + |t|)^{n/4 - 1/12n^4}.}$$

Uniformity in π

For π on GL_n of level one (for simplicity), $L(\pi, s)$ of degree n satisfies a functional equation under $s \mapsto 1 - s$ involving

$$\Gamma\left(\frac{s + i\lambda_1}{2}\right) \cdots \Gamma\left(\frac{s + i\lambda_n}{2}\right) L(\pi, s)$$

for some complex numbers $\lambda_1, \ldots, \lambda_n$.
Analytic conductor:

$$\mathrm{Cond}(\pi, t) := \prod_{j=1}^{n} (1 + |\lambda_j + t|).$$

Convexity bound can be made uniform in π (Molteni 2002):

$$L(\pi, 1/2 + it) \ll_{n,\varepsilon} \mathrm{Cond}(\pi, t)^{1/4+\varepsilon}.$$

Subconvexity for standard L-functions assuming "uniform parameter growth"

We have the following generalization of Theorem 1:

Theorem 2 (N, arXiv, October 2021)

Let π be on GL_n and of level one (for simplicity).
Assume the "uniform parameter growth" condition

$$\boxed{\frac{T}{2022} \leqslant |\lambda_1 + t|, \ldots, |\lambda_n + t| \leqslant T} \qquad \text{for some } T \geqslant 1.$$

Then

$$\boxed{L(\pi, 1/2 + it) \ll_n \mathrm{Cond}(\pi, t)^{1/4 - 1/12n^5}.}$$

Important open problem: remove uniform growth assumption.
Application to "strong AQUE."

Theorem 1 is the subject of the third paper in a "series:"

1. *The orbit method and analysis of automorphic forms*
   - with A. Venkatesh; arXiv, May 2018; Acta Math. 2021
   - asymptotics for averages of families of L-functions $L(\pi \times \sigma, 1/2)$ on $\mathrm{SO}_{n+1} \times \mathrm{SO}_n$ or $\mathrm{U}_{n+1} \times \mathrm{U}_n$.
2. *Spectral aspect subconvex bounds for* $\mathrm{U}_{n+1} \times \mathrm{U}_n$
   - arXiv, Dec 2020; submitted
   - subconvex bounds for L-functions

     $$L(\pi \times \sigma, 1/2)$$

     attached to automorphic forms on $\mathrm{U}_{n+1} \times \mathrm{U}_n$ with U_n anisotropic, so that $\mathrm{U}_n(\mathbb{Z}) \backslash \mathrm{U}_n(\mathbb{R})$ is compact.
   - S. Marshall: Mar 2018 announcement of "special case" (well-spaced parameters, depth aspect)
3. *Bounds for standard L-functions*
   - arXiv, Oct 2021
   - e.g., removes compactness assumption from paper 2, specializes σ to be the Eisenstein series for which

     $$L(\sigma, s) = \zeta(s)^n, \quad L(\pi \times \sigma, s) = L(\pi, s)^n.$$

Discussion of methods

- Main tool for rigorous study of L-functions: integral representations involving automorphic forms.
- Basic example:

$$\underbrace{\int \theta(iy) y^{s/2} \frac{dy}{y}}_{\text{integral of an automorphic form}} \doteq \underbrace{2\pi^{-s/2}\Gamma(s/2)}_{\text{"special function"}} \underbrace{\zeta(s)}_{\text{value of an } L\text{-function}} .$$

- $\zeta(s)^2$ comes from the GL_2 modular form

$$E(x+iy) = \sum_{n=1}^{\infty} \tau(n) \cos(2\pi nx) \sqrt{y} K_0(2\pi ny) + (\cdots),$$

where K_0: Bessel function ("GL_2 special function").

$$\int E(iy) y^s \frac{dy}{y} = \pi^{-s} \Gamma\left(\frac{s}{2}\right)^2 \zeta(s)^2.$$

Reason: special function integral evaluation

$$\boxed{\int K_0(2\pi y) y^s \frac{dy}{y} = \pi^{-s} \Gamma\left(\frac{s}{2}\right)^2.}$$

- Analogues of this integral evaluation in higher degree are more complicated.

Automorphic forms on GL_n are functions

$$v : \mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R}) \to \mathbb{C}$$

satisfying certain "eigenfunction" or "irreducibility" conditions. They may be organized into representations: irreducible spaces $\pi \subseteq C^\infty(\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R}))$.

For $v \in \pi$ on GL_{n+1} and $u \in \sigma$ on GL_n,

$$\int_{\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})} \text{restriction}(v) \cdot u = L(\pi \times \sigma, 1/2) \times (\text{special function})_{v,u}.$$

To study $L(\pi \times \sigma, 1/2)$ this way,[4] we must understand

- the special functions, and
- the integrals of automorphic forms

for suitable v and u. Remarks:

- $(\text{special function})_{v,u}$ is the integral of the product of archimedean Whittaker functions attached to v and u.
- If σ is an Eisenstein series with trivial parameters, then

$$L(\pi \times \sigma, s) = L(\pi, s)^n.$$

[4] following Bernstein–Reznikov, Michel–Venkatesh

The proofs of our recent results concerning L-functions involve

1. a systematic approach to the "special functions" analysis required by those proofs, based on the orbit method and microlocal analysis, and
2. additional "global" arguments:
   - (N–V 2018): Ratner theory
   - (N 2020, 2021): analysis of a relative trace formula, invariant-theoretic problems, ...

Goal of the lectures

- Develop the orbit method in analytic form as a microlocal calculus for Lie group representations, sharp up to ε's.
- Describe "special function asymptotics" relevant for integral representations of L-functions.
- Invariant-theoretic issues: "stability," "transversality."

Today: a very quick survey of the main points concerning each of these and how they fit together.

Microlocalized vectors and coadjoint orbits

- $\pi \hookrightarrow C^\infty(\Gamma \backslash G)$, $G = \mathrm{GL}_n(\mathbb{R})$.
- $\mathcal{O}_\pi := \{n \times n$ matrices with eigenvalues $\lambda_1, \dots, \lambda_n\}$.
- $\mathcal{O}_\pi \ni \tau \rightsquigarrow$ a class of vectors $v \in \pi$ "microlocalized at τ:"

  $$\pi(\exp(x))v \approx e^{i\,\mathrm{trace}(x\tau)}v$$

  for all $x \in \mathfrak{g}$ in a suitable small neighborhood of the origin.
- Can define $\omega_\tau \in C_c^\infty(\mathrm{GL}_n(\mathbb{R}))$ such that

  $$\pi(\omega_\tau) \approx \text{ projection onto } v \text{ satisfying above conditions.}$$

- $\pi(\omega_\tau)$ behaves like a rank ≈ 1 idempotent projector
- $\mathcal{F}_\tau := \{\pi : \pi(\omega_\tau) \not\approx 0\}$ is a short family

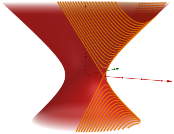
Similar discussion applies more generally to any reductive group G.

Example

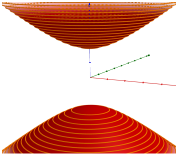
$G = \mathrm{PGL}_2(\mathbb{R}) \cong \mathrm{SO}(1,2)$

$\mathfrak{g}^\wedge \ni \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}$

- one-sheeted hyperboloid
  $\mathcal{O}^+(r) = \{x^2 + y^2 - z^2 = r^2\}$



- two-sheeted hyperboloid
  $\mathcal{O}^-(k) = \{x^2 + y^2 - z^2 = -k^2\}$



Tempered irreducible representations:

- principal series $\pi(r, \varepsilon)$
- discrete series $\pi(k)$ for $k \in \mathbb{Z}_{\geqslant 1}$

$$\mathcal{O}_{\pi(r,\varepsilon)} = \mathcal{O}^+(r), \quad \mathcal{O}_{\pi(k)} = \mathcal{O}^-(k - 1/2)$$

Special function asymptotics

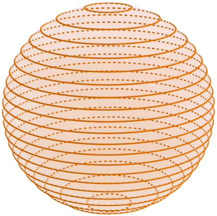
1. If v and u are microlocalized, say at $\tau = \begin{pmatrix} \tau_H & * \\ * & * \end{pmatrix} \in \mathcal{O}_\pi$ and $\eta \in \mathcal{O}_\sigma$, then [NV, N]

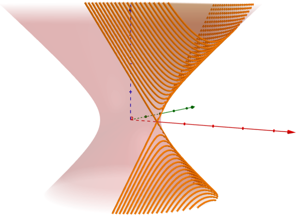
$$(\text{special function})_{v,u} \approx \begin{cases} 0 & \text{if } \tau_H + \eta \not\approx 0, \\ T^{-n^2/4} & \text{if } \tau_H + \eta \approx 0. \end{cases}$$

2. $\mathrm{GL}_n(\mathbb{R})$ acts on $\{\xi \in \mathcal{O}_\pi : -\xi_H \in \mathcal{O}_\sigma\}$ by conjugation, often simply-transitively ($\rightsquigarrow$ stability in the sense of geometric invariant theory).

Similar discussion applies to many strong Gelfand pairs (G, H), i.e., $(\mathrm{U}_{n+1}, \mathrm{U}_n)$ or $(\mathrm{SO}_{n+1}, \mathrm{SO}_n)$ over $\mathbb{R}$ or $\mathbb{C}$.

Some pictures of $\{\xi \in \mathcal{O}_\pi : -\xi_H \in \mathcal{O}_\sigma\}$ for $(G, H) = (\mathrm{SO}(3), \mathrm{SO}(2))$ and $(\mathrm{PGL}_2(\mathbb{R}), \mathrm{GL}_1(\mathbb{R}))$:





Given a short family $\mathcal{F} \subset \{\pi\text{'s on } \mathrm{PGL}_{n+1}\}$ and σ on GL_n, we choose $\tau \asymp T$ with $\tau \in \mathcal{O}_\pi$, $-\tau_H \in \mathcal{O}_\sigma$, so $\mathcal{F} \approx \mathcal{F}_\tau$, and let $v_\pi \in \pi$ and $u \in \sigma$ be microlocalized at τ and $-\tau_H$. Then for $x, y \in \mathrm{PGL}_{n+1}(\mathbb{R})$,

$$
\begin{aligned}
k(x,y) &:= \sum_{\gamma \in \mathrm{PGL}_{n+1}(\mathbb{Z})} \omega_\tau(x^{-1}\gamma y) \\
&= \sum_{\pi} \sum_{v \in \mathcal{B}(\pi)} \pi(\omega_\tau) v(x) \overline{v(y)} \\
&\approx \sum_{\pi \in \mathcal{F}} v_\pi(x) \overline{v_\pi(y)},
\end{aligned}
$$

hence

$$
\begin{aligned}
&\int_{x,y \in \mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n(\mathbb{R})} k(x,y) u(x) \overline{u(y)} \, dx \, dy \\
&\quad \approx T^{-n^2/2} \sum_{\pi \in \mathcal{F}} \left| L(\pi \otimes \sigma, \tfrac{1}{2}) \right|^2 .
\end{aligned}
$$

In summary,

$$T^{-n^2/2} \sum_{\pi \in \mathcal{F}} \left| L(\pi, \tfrac{1}{2}) \right|^{2n}$$
$$\approx \int_{x,y \in \mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n(\mathbb{R})} u(x)\overline{u(y)} \sum_{\gamma \in \mathrm{PGL}_{n+1}(\mathbb{Z})} \omega_\tau(x^{-1}\gamma y)\, dx\, dy.$$

Requires asymptotic evaluation.

- The $\gamma \in \mathrm{GL}_n(\mathbb{Z})$ contribute a "main term," addressed via amplification.
- Estimating $\sum_{\gamma \in \mathrm{PGL}_{n+1}(\mathbb{Z}) - \mathrm{GL}_n(\mathbb{Z})}$ uses:
  - A linear-algebraic fact concerning τ (consequences: "transversality," "bilinear forms estimate").
  - A local L^2 growth bound for u.