On the classical Turán problem for 3-graphs

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Turán numbers

$\alpha(H) :=$ *independence number* of hypergraph H

An n-vertex r-graph is a *Turán* (n,k,r)*-system* if $\alpha(H) < k$.

$$T(n,k,r) \; := \; \min\{|E(H)| : \; |V(H)| = n, \; \alpha(H) < k\}$$

$\binom{n}{r} - T(n,k,r)$ is the largest number of edges in an n-vertex r-graph that does not contain a complete k-vertex subgraph.

Mantel (1907): $T(n,3,2)$

Turán (1941): $T(n,k,2)$

optimal construction: $k-1$ disjoint cliques of almost equal size

Turán's conjecture. The optimal construction for $T(n,(r-1)s+1,r)$ is s disjoint cliques of almost equal size.

Turán's conjecture

Conjecture. The optimal construction for $T(n,(r-1)s+1,r)$ is s disjoint cliques of almost equal size.

Does not hold for $r \geq 4$, even asymptotically.

$$T(n,7,4) \overset{?}{\approx} \frac{1}{8} \cdot \binom{n}{4}$$

$$T(n,7,4) \leq \frac{443}{640} \cdot \frac{1}{8} \cdot \binom{n}{4}$$

What about $r = 3$?

Turán's conjecture for $T(n, 2s+1, 3)$

Conjecture. The optimal construction for $T(n, 2s+1, 3)$ is s disjoint cliques of almost equal size.

No counterexample is known when $n \equiv 0 \mod s$.

We will concentrate on $T(n, 5, 3)$.

Turán's conjecture for $T(n,5,3)$

Conjecture. Two disjoint cliques of almost equal size give an optimal construction for $T(n,5,3)$.

For odd n, there are counterexamples.

J. Surányi (1971):
AG(2,3) yields $T(9,5,3) \leq 12 < 14 = \binom{5}{3} + \binom{4}{3}$.

Collinear triples of points in PG(2,3) yield:
$T(13,5,3) \leq 13 \cdot 4 = 52 < 55 = \binom{7}{3} + \binom{6}{3}$.

Kostochka's construction:
$T(8m+1,5,3) \leq \frac{4}{3}m(2m-1)(8m+1) < \binom{4m+1}{3} + \binom{4m}{3}$.

The exact values of $T(n,5,3)$ are known for $n \leq 17$ (Markström).

A plausible conjecture for $T(2m+1,5,3)$

Suppose Turán conjecture for $n = 2m$ holds: $T(2m,5,3) = 2\binom{m}{3}$.

Since $T(n,k,r)/\binom{n}{r}$ is nondecreasing in n,

$$\frac{T(2m+1,5,3)}{\binom{2m+1}{3}} \;\geq\; \frac{T(2m,5,3)}{\binom{2m}{3}},$$

$$\begin{aligned} T(2m+1,5,3) \;&\geq\; \left\lceil \frac{2m+1}{2m-2} \cdot 2\binom{m}{3} \right\rceil \\ &=\; \begin{cases} \frac{1}{6}(2m+1)m(m-2) \text{ if } m \text{ is even;} \\ \\ \frac{1}{6}(2m-3)(m-1)(m+1) \text{ if } m \text{ is odd.} \end{cases} \end{aligned}$$

Our aim is to prove the matching upper bound on $T(2m+1,5,3)$.

Monochromatic triangles in 2-edge-colored K_n

Goodman (1959):

$M(n)$ is the minimum # of monochromatic triangles in K_n

M is # of monochromatic triangles
R is # of rainbow triangles
P is # of monochromatic 2-edge paths

Then $M + R = \binom{n}{3}$, $P = 3M + R$, so $M = \frac{1}{2}\left(P - \binom{n}{3}\right)$.

$x_j :=$ # black edges $-$ #white edges incident to vertex j

# of monochromatic 2-edge paths with j as the middle vertex is
$\binom{\frac{1}{2}(n-1+x_j)}{2} + \binom{\frac{1}{2}(n-1-x_j)}{2} = \frac{1}{4}(n^2 - 4n + 3 + x_j^2)$

$$M = \frac{1}{8}\sum_{j=1}^{n}(n^2-4n+3+x_j^2) - \frac{1}{2}\binom{n}{3} = \frac{1}{24}n(n-1)(n-5) + \frac{1}{8}\sum_{j=1}^{n} x_j^2.$$

Monochromatic triangles in 2-edge-colored K_n

$$M(n) \;=\; \frac{1}{24}n(n-1)(n-5) \;+\; \frac{1}{8}\min_{x_1,\ldots,x_n}\;\sum_{j=1}^{n} x_j^2.$$

If n is even, x_j must be odd.
The minimum is attained when $x_1^2 = \ldots = x_n^2 = 1$.

If $n \equiv 1 \mod 4$, the minimum is attained when $x_1 = \ldots = x_n = 0$.

If $n \equiv 3 \mod 4$, the number of edges in K_n is odd.
The minimum is attained when $x_1 = \pm 2$ and $x_2 = \ldots = x_n = 0$.

$$M(n) = \begin{cases} 2\binom{m}{3} \;\text{ if }\; n = 2m; \\ \\ \frac{1}{6}(2m+1)m(m-2) \;\text{ if }\; n = 2m+1,\; m \;\text{ is even;} \\ \\ \frac{1}{6}(2m-3)(m-1)(m+1) \;\text{ if }\; n = 2m+1,\; m \;\text{ is odd.} \end{cases}$$

Monochromatic triangles as a Turán $(n, 5, 3)$-system

A coloring is called *balanced* when $x_1 = \ldots = x_n = 0$.

The only coloring of K_5 without monochromatic triangles
is the balanced one (edges of the same color form a 5-cycle).

If in a coloring of K_n,
no 5 vertices induce a balanced coloring of K_5,
then the monochromatic triangles form a Turán $(n, 5, 3)$-system.

Example 1. Take two groups of vertices of size m. Color edges inside each group with black, color the edges connecting the groups with white. The monochromatic triangles form two disjoint cliques.

Example 2. Consider 9 vertices of a 3×3 grid.
Color all horizontal and vertical segments with black,
and all diagonal segments with white.
The 12 monochromatic triangles (6 black and 6 white)
form the 12 lines of AG(2,3).

Monochromatic triangles as Turán (5,3)-systems

We are going to prove $T(n,5,3) \leq M(n)$ for $n = 2m+1 \geq 17$, $n \neq 27$. (Recall that the values of $T(n,5,3)$ for $n \leq 17$ are known).

To do so, we will strengthen Goodman's result by constructing an (almost) balanced coloring of K_n that does not induce a balanced coloring of K_5.

Such a coloring does not exists for $n = 5, 7, 11, 13, 15$.

For any $n \geq 1$, with the exception of $n = 13$, the currently best upper bound on $T(n,5,3)$ can be achieved by a system of monochromatic triangles in K_n.

Avoiding 3 edges on 4 vertices

When 4 vertices in a 3-graph span exactly 3 edges,
we call it a $(4,3)$-*configuration*.

It is easy to see that a system of monochromatic triangles can not produce a $(4,3)$-configuration.

The system of collinear triples in PG(2,3) does not contain such a configuration either.

Let $B(n,k)$ denote the currently best upper bound for $T(n,k,3)$.

In particular, $B(n,5) = M(n) + 1$ if $n = 5, 7, 11, 15, 27$, and $B(n,5) = M(n)$ for all other n.

Then for any n, there exists a Turán $(n,5,3)$-system of size $B(n,5)$ that does not contain a $(4,3)$-configuration.

In fact, for any k and any n, there exists a Turán $(n,k,3)$-system of size $B(n,k)$ that does not contain a $(4,3)$-configuration.

Avoiding 3 edges on 4 vertices

Razborov (2010), Pikhurko (2011):

For large n, the minimum size of a Turán $(n, 4, 3)$-system that does not contain a $(4, 3)$-configuration is the same as the conjectured value of $T(n, 4, 3)$.

Problem. Prove that every Turán $(n, 5, 3)$-system without a $(4, 3)$-configuration has at least $M(n)$ edges.

Regular graphs without induced 5-cycles

Instead of coloring the edges of K_n with black and white, we may consider the subgraph formed by the black edges.

Theorem 1. For $n \equiv 1 \mod 4$, $n \neq 5, 13$, there exists a $\frac{n-1}{2}$-regular n-vertex graph without induced 5-cycles.

We call a graph *almost d-r*egular if all vertices except one have degree d, and the remaining vertex has degree $d \pm 1$.

Theorem 2. For $n \equiv 3 \mod 4$, $n \neq 7, 11, 15, 27$, there exists an almost $\frac{n-1}{2}$-regular n-vertex graph without induced 5-cycles.

Regular graphs without induced 5-cycles

We will try to enlarge the problem and find all pairs (n, d) such that there exists an (almost) d-regular n-vertex graph without induced 5-cycles.

Lemma 3. Let n be even. For any d such that $0 \leq d \leq n-1$, there exists a d-regular n-vertex graph without induced 5-cycles.

Lemma 4. Let n be odd. For any even d such that $0 \leq d \leq n-1,\ d \neq \frac{n-1}{2}$, there exists a d-regular n-vertex graph without induced 5-cycles.

Lemma 5. Let $n \equiv 3 \bmod 4$. For any odd d such that $1 \leq d \leq n-2,\ d \neq \frac{n-1}{2}$, there exists an almost d-regular n-vertex graph without induced 5-cycles.

Regular graphs without induced 5-cycles

Lemma 3. Let n be even. For any d such that $0 \leq d \leq n-1$, there exists a d-regular n-vertex graph without induced 5-cycles.

Proof. Let $n = 2m$. The induction basis at $m = 1$ is trivial.
Let $m > 1$. We may assume $d \leq m-1$.
If $d = m-1$, take the union of two disjoint K_m.
If $d \leq m-2$, let k be an even number equal either $d+1$ or $d+2$.
Then $n-k \geq 2m-(d+2) \geq m > d$.
By the induction hypothesis, there exists
an $(n-k)$-vertex d-regular graph without induced 5-cycles,
as well as a k-vertex d-regular graph without induced 5-cycles.
Take the union of their disjoint copies. □

Regular graphs without induced 5-cycles

Lemma 6. Suppose graphs G and H do not contain induced 5-cycles. Blow up a vertex of G and replace it with a copy of H. Then the resulting graph does not contain induced 5-cycles.

Plan of proof of Theorem 1:

− take a small graph G without induced 5-cycles;

− blow up each vertex and replace it with its own regular graph without induced 5-cycles;

− select the orders and degrees of these graphs to ensure that the resulting graph is $\frac{n-1}{2}$-regular.

We elect G to be the smallest $\frac{n-1}{2}$-regular n-vertex graph without induced 5-cycles, which is (at $n = 9$) the 3×3 grid graph.

Regular graphs without induced 5-cycles

The 3×3 grid graph is a 4-regular graph with 9 vertices.

$$\begin{array}{ccccc}
(n_1, d_1) & — & (n_2, d_2) & — & (n_3, d_3) \\
| & & | & & | \\
(n_4, d_4) & — & (n_5, d_5) & — & (n_6, d_6) \\
| & & | & & | \\
(n_7, d_7) & — & (n_8, d_8) & — & (n_9, d_9)
\end{array}$$

We have 18 variables and 1+9=10 equations:

$$n_1 + \ldots + n_9 = n,$$

$$d_1 + n_2 + n_3 + n_4 + n_7 = \frac{n-1}{2}, \quad \ldots$$

$$n_j = \frac{n}{9}, \quad d_j = \frac{1}{2}\left(\frac{n}{9} - 1\right).$$

Restrictions: $0 \leq d_j \leq n_j - 1$,

If n_j is odd, then d_j is even, $d_j \neq \frac{n_j - 1}{2}$.

Regular graphs: proof of Theorem 1

$n = 8k + 1$

$$\begin{array}{ccc} (1,0) & (k,k-1) & (k,k-1) \\ (k,k-1) & (k,0) & (k,0) \\ (k,k-1) & (k,0) & (k,0) \end{array}$$

Now we may relax restrictions and allow $(n_j, d_j) = (8k+1, 4k)$.

$n = 8k + 5,\ \ k \geq 2\ :\quad$ 7 cases, depending on $n \bmod 72$.

$n = 72r + 5$

$$\begin{array}{ccc} (8r+1, 4r+2) & (8r+1, 4r-2) & (8r+1, 4r-2) \\ (8r-1, 4r) & (8r+1, 4r) & (8r+1, 4r) \\ (8r-1, 4r) & (8r+1, 4r) & (8r+1, 4r) \end{array}$$

Almost regular graphs without induced 5-cycles

$n = 4k + 3, \;\; k \neq 1, 2, 3, 6.$

$$\begin{array}{ccc} ((n_1, d_1)) & (n_2, d_2) & (n_3, d_3) \\ (n_4, d_4) & (n_5, d_5) & (n_6, d_6) \\ (n_7, d_7) & (n_8, d_8) & (n_9, d_9) \end{array}$$

"$((n_1, d_1))$" denotes almost d_1-regular n_1-vertex graph without induced 5-cycles

$n = 36r + 7$

$$\begin{array}{ccc} ((4r+3, 2r-1)) & (4r+2, 2r-2) & (4r+2, 2r-2) \\ (4r, 2r) & (4r, 2r+1) & (4r, 2r+1) \\ (4r, 2r) & (4r, 2r+1) & (4r, 2r+1) \end{array}$$

What about n=27 ?

There is a system of monochromatic triangles that yields $T(27,5,3) \leq M(27)+1$.

3 possible outcomes:

- $T(27,5,3) = M(27)+1$ (similar to $n = 5, 7, 11, 15$).
- Every almost 13-regular 27-vertex graph contains an induced 5-cycle, but there exists non-coloring based Turán $(27,5,3)$-system of size $M(27)$ (similar to $n = 13$), so $T(27,5,3) = M(27)$.
- There exists an almost 13-regular 27-vertex graph without induced 5-cycles, so $T(27,5,3) = M(27)$.

“Corrected” Turán’s conjecture for $T(n, 5, 3)$

Conjecture.

$$T(n, 5, 3) = \begin{cases} M(n) + 1 \text{ for } n = 5, 7, 11, 15, 27; \\ M(n) \text{ for all other } n. \end{cases}$$

$T(n, k, 3)$ construction for arbitrary k

$k - 1$ cyclically ordered cliques $B_0, ..., B_{k-2}$
with additional triples $\{x, y, z\}$ where $x, y \in B_j$, $z \in B_{j+1}$,
$j \in \mathbb{Z}_{k-1}$.

Conjecture. This construction is optimal for $n \equiv 0 \mod k - 1$.

When $k = 4, 6$ this construction seems to be optimal
for all values of n.

The exact values of $T(n, 4, 3)$, $T(n, 6, 3)$ are known for $n \leq 18$
(Markström).

Our results for $T(2m + 1, 5, 3)$ yield improvements for all $k \geq 7$.

Fon-Der-Flaass' construction for $T(n, 4, 3)$

Fon-Der-Flaass (1988) : Take an n-vertex oriented graph Γ and construct a 3-graph H with the same vertex set.

Vertices a, b, c form an edge in H if one of the two conditions met:

1). (a, b), (a, c) are arcs of Γ.

2). a, b, c induce at most one arc in Γ.

If no 4 vertices in Γ induce an oriented 4-cycle, then H is a Turán $(n, 4, 3)$-system.

Kostochka found 2^{m-2} nonisomorphic constructions for $T(3m, 4, 3)$.

For $m \leq 6$, there are no other constructions with the same number of edges.

Fon-Der-Flaass' construction allows to recover all of the Kostochka's constructions.

Generalized Fon-Der-Flaass' construction

Color arcs of an n-vertex oriented graph Γ with $k-1$ colors.

Vertices a, b, c form an edge in 3-graph H
if one of the two conditions met:

1). (a, b), (a, c) are arcs of the same color in Γ.

2). a, b, c induce at most one arc in Γ.

If $2k$ vertices do not contain an edge in H,

then they induce an orgraph in Γ where

each vertex has indegree 1 and outdegree 1 in each color.

If Γ does not have such induced subgraphs,

then H is a Turán $(n, 2k, 3)$-system.

This $(k-1)$-color construction allows to recover the standard "cyclical" construction for $T(n, 2k, 3)$.

$T(17, 8, 3) = 31$ can not be recovered by the 3-color construction.

Thank you!